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in mind, however, that the singularities thus brought out are not the point singularities, but the corresponding line singularities: double tangents, isolated double tangents and inflexional tangents. The equations become

$$\sigma x = \Phi(\lambda) \equiv \chi(\lambda)\psi'(\lambda) - \psi(\lambda)\chi'(\lambda),$$
$$\sigma y = \Psi(\lambda) \equiv \varphi(\lambda)\chi'(\lambda) - \chi(\lambda)\varphi'(\lambda),$$
$$\sigma z = \mathrm{X}(\lambda) \equiv \psi(\lambda)\varphi'(\lambda) - \varphi(\lambda)\psi'(\lambda).$$

If Φ, Ψ, X have no common factor, the order of the curve is $2[2(n-1)-\kappa-1]$, this being the degree of Φ, Ψ, X; if they have ι factors in common, corresponding to ι inflexions, the order is $2[2(n-1)-\kappa-1]-\iota$. [*Cf.* the parallel statement, (4) to (4′), above.] The order of the curve is n; hence

$$n = 2[2(n-1)-\kappa-1]-\iota,$$

$$2\kappa + \iota = 3(n-2). \tag{5}$$

(5) indicates that the number of inflexions is odd or even according as n is odd or even; it also yields an upper limit for the number of inflexions,

$$\iota \leqq 3(n-2), \tag{6}$$

or of cusps,

$$\kappa \leqq \tfrac{3}{2}(n-2). \tag{7}$$

Wieleitner (*loc. cit.*) obtains this result (7) by means of Plücker's formulas.

When n is odd, the right-hand member of (7) is not an integer; but in this case there is at least one inflexion, and $\frac{3(n-2)-1}{2}$, which is the integral part of the fraction $\frac{3}{2}(n-2)$, gives the greatest possible number of cusps.

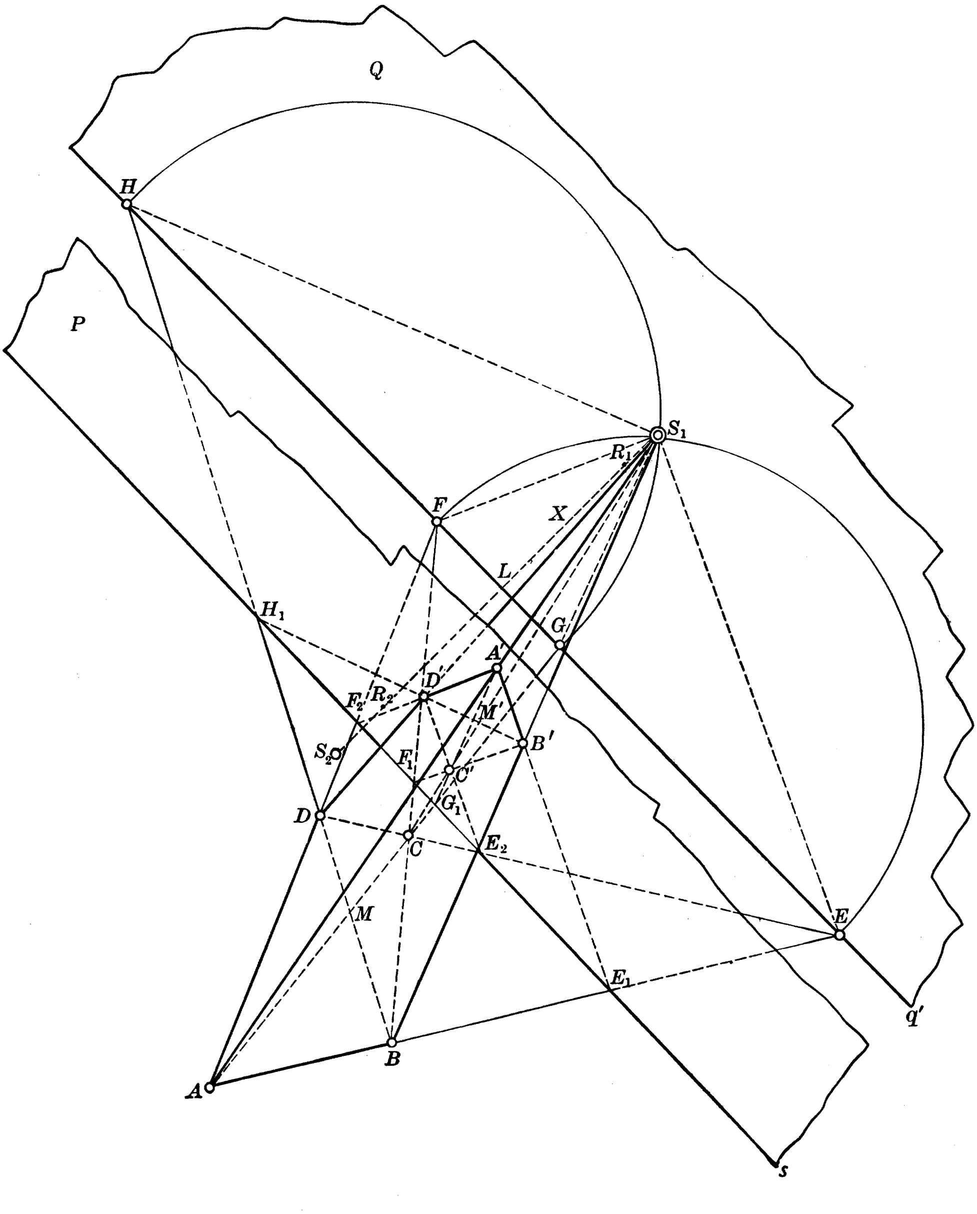
For $n > 4$, the maximum number of cusps is less than the total number of double points.

A PROBLEM IN PERSPECTIVE.

By ARNOLD EMCH, University of Illinois.

1. It is well known that any proper plane quadrangle, *i. e.*, a figure determined by four coplanar points of which no three are collinear, may be considered in an infinite number of ways as the perspective of a square, so that to the vertices of the square correspond in a certain order the vertices of the quadrangle. In such a perspective the points within the surface determined in the ordinary sense by the square and the quadrangle do not necessarily have to correspond to each other. For the sake of clearness of representation the figure shows a case of ordinary pictorial perspective in which also the ordinary surfaces correspond to each other. This does not involve loss of generality.

The purpose of the present note is to show how this problem may be treated in an extremely simple manner without going beyond the rudiments of elementary solid geometry and to emphasize, from a didactic standpoint, the importance of perspective as an introduction to projective geometry.



2. In a plane R assume any proper quadrangle $ABCD$ and let the joins of A and B, and of C and D meet at E, those of B and C, and of A and D at F.

Let q' be the join of E and F. Outside of R assume a point S_1 as the vertex of a pyramid having $ABCD$ as a base. Join S_1 to E and F; S_1E is the intersection of the planes of the faces S_1AB and S_1CD of the pyramid, S_1F that of S_1CB and S_1AD extended. Now any plane parallel to S_1E and not containing S_1 cuts the faces S_1AB and S_1CD in two parallel lines. Likewise a plane parallel to S_1F cuts S_1AD and S_1BC in two parallel lines.

Consequently every plane P parallel to the plane Q determined by S_1E and S_1F cuts the pyramid in a parallelogram $A'B'C'D'$, such that $A'B' \parallel C'D' \parallel S_1E$ and $A'D' \parallel B'C'' \parallel S_1F$. If we choose S_1 on a circle with EF as a diameter, then $S_1E \perp S_1F$, and $A'B'C'D'$ becomes a rectangle.

Let G and H be the intersections of the third pair of sides AC and BD of the quadrangle with q'. The plane determined by A, C and S_1 cuts the plane P in the join of A' and C', so that $A'C' \parallel S_1G$. Likewise, the plane determined by B, D, and S_1 cuts P in the join of B' and D' so that $B'D' \parallel S_1H$. Now a square may be defined as a rectangle with perpendicular diagonals. Hence, when S_1 also lies on a circle with GH as a diameter, so that not only $S_1E \perp S_1F$, but also $S_1G \perp S_1H$, then in the rectangle $A'B'C'D'$ the diagonals $A'C'$ and $B'D'$ are at right angles and $A'B'C'D'$ becomes a square. From this is seen that not every quadrangular pyramid admits of square sections. In order that this be the case, when the base $ABCD$ is arbitrarily given, the vertex S_1 must satisfy a certain condition, which according to the foregoing construction may be stated as follows:

Through any of three diagonals q' of the quadrangle pass any plane Q and in it draw two circles with the segment between the two diagonal points in q' as a diameter of one of the circles and with the segment determined on q' by the third pair of sides of the quadrangle as a diameter of the other circle. The intersections S_1 and S_2 of these two circles are the vertices of two pyramids with the given quadrangle as a base, such that every plane parallel to Q cuts these pyramids in squares.

In every plane Q through q' there are two such circles, and when Q revolves about q', these circles generate two spheres with their centers on q', and which intersect in a circle X, the locus of the vertices S_1 of a set (X) of pyramids with the common base $ABCD$, and admitting of square plane section. Every plane parallel to the plane determined by S_1 and q' cuts the pyramid $S_1 \cdot ABCD$ in a square.

As the complete quadrangle $ABCD$ has three diagonals, which we denote by q', r', s', there are evidently three sets of pyramids having the given quadrangle as a base and admitting of plane square sections. The loci of the vertices of these pyramids are three circles which we denote in the same order by X, Y, Z.

In all previous statements it is assumed that the three diagonals q', r', s' exist in the Euclidean sense, although, under certain modifications, they may be extended to cases in which one of the diagonals is infinitely distant, *i. e.*, in which the quadrangle is a parallelogram. The segments EF and GH on the diagonal q' and the corresponding segments on r' and s' are the diameters of the three couples of spheres that intersect in the three circles X, Y, Z. These results may be stated in the form of a theorem as follows:

THEOREM. *The locus of the vertices of all pyramids that have a proper coplanar quadrangle with three existing diagonals as a common base and which admit of square plane sections consists of three circles whose centers lie in, and whose planes are orthogonal each for each to, the three diagonals. Each of these circles is the intersection of two spheres with the two segments, like EF and GH, on each diagonal as diameters. If S_1 is a point on any of those circles attached to one of the diagonals, then any plane parallel to the plane determined by S_1 and this diagonal cuts the pyramid $S_1 \cdot ABCD$ in a square.*

3. For the sake of clearness the figure shows the construction for one diagonal, q', only. Any corresponding lines of the quadrangle and the square, like AB and $A'B'$ prolonged, meet in a point of s, the intersection of the planes R and P. The construction is still valid for the intersections R_1 and R_2 of X with the plane R. For these vertices the faces of the pyramid coincide with R. The reason for the validity in these limiting cases can easily be explained but, for the sake of brevity, will be omitted. In the language of central projection or perspective P may be considered as the plane of the object, the square $A'B'C'D'$; R as the picture-plane, S_1 as the center, s as the axis, q' as the vanishing-line of the perspective. E and F are the vanishing points of the two pairs of parallel sides of the square, and the quadrangle $ABCD$ is the perspective of the square.

4. If through S_1 we pass a plane parallel to R, it will cut P in a line t' whose perspective in a projective sense is the infinite line t of R.

By means of this perspective we can easily derive all the polar-properties of a conic from those of a circle. For instance, if we circumscribe a circle K' to the square and construct the pole T' of t' with respect to K', then T' will be projected into the pole T of the infinite line t of R with respect to the perspective K of K', which is a conic circumscribed to the quadrangle $ABCD$. By definition T is now the center of the conic K. According to the disposition in the figure this conic is an ellipse. To avoid the overloading of the figure this part of the construction, which would be simple enough, has been omitted. It follows also without difficulty that two conjugate polars of the circle through T' project into two conjugate diameters of the conic.

Thus we have reached the path by which in the historic development of projective geometry the theory of conics was derived from that of the circle.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

The Elements of Non-Euclidean Plane Geometry and Trigonometry. By H. S. CARSLAW. Longmans, Green and Co., London, 1916. xii + 179 pages. $1.50.

In this work Professor Carslaw has given an excellent introduction to the classical non-euclidean geometries in the plane, an introduction of sufficient